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MULTIPLE SCATTERING OF WAVES BY A VOLUME DISTRIBUTION OF PARALLEL CYLINDERS

by

VIC TWERSKY

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ABSTRACT

We consider the multiple scattering of a plane electromagnetic or acoustic wave by a uniform distribution of cylinders lying parallel to the z-axis in the range $x = 0$ to d, $y = -\infty$ to ∞ (i.e., in an infinite slab). A closed-form representation for the average wave function is derived; then a heuristic procedure is used to obtain the incoherent scattering. Finally certain energy theorems are stated and it is shown that the results of this paper are in accord with the energy principle.

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Procedure

In a previous paper[1] we derived a formal multiple scattering solution of the problem of a plane wave incident on an arbitrary configuration of parallel cylinders. In the present paper we average this function over a uniform distribution of identical cylinders lying in the range x = 0 to d, y = $-\infty$ to ∞ (i.e., an infinite slab as in Fig. 1); all axes are parallel to the z-axis. We derive a closed-form representation for the average wave function; then we follow a heuristic procedure to obtain the 'incoherent' scattering. Finally certain energy theorems are stated and it is shown that our results are in accord with the energy principle.

For the case of isotropic scatterers our wave function is essentially of the form obtained previously by Reiche[2] and Foldy[3]; for non-isotropic scatterers our wave function differs from that of Lax[4] in that the propagation constant depends on the angle of incidence.

The Wave Scattered by an Arbitrary Configuration

The two-dimensional multiple scattering of a plane electromagnetic or acoustic wave by an arbitrary configuration of parallel cylinders may be formulated as follows[1] : We seek a function $\bar{\Psi}(\underline{r})$ such that

$$(\nabla^2 + k^2)\,\bar{\Psi} = 0 \quad , \quad k = 2\pi/\lambda \quad ,$$

$$\bar{\Psi} = \psi_i + \sum \psi_s \,(|\underline{r} - \underline{r}_s|) \quad , \tag{1}$$

$$\psi_i = \text{plane wave} \quad , \quad \psi_s \to 0 \text{ as } |\underline{r} - \underline{r}_s| \to \infty \ ;$$

this function should also satisfy prescribed boundary conditions at the surfaces $|\underline{r} - \underline{r}_s| = a$ of the cylinders. Here $\underline{r}(r, \theta)$ locates the observation point and $\underline{r}_s$ locates cylinder s; see Fig. 1.

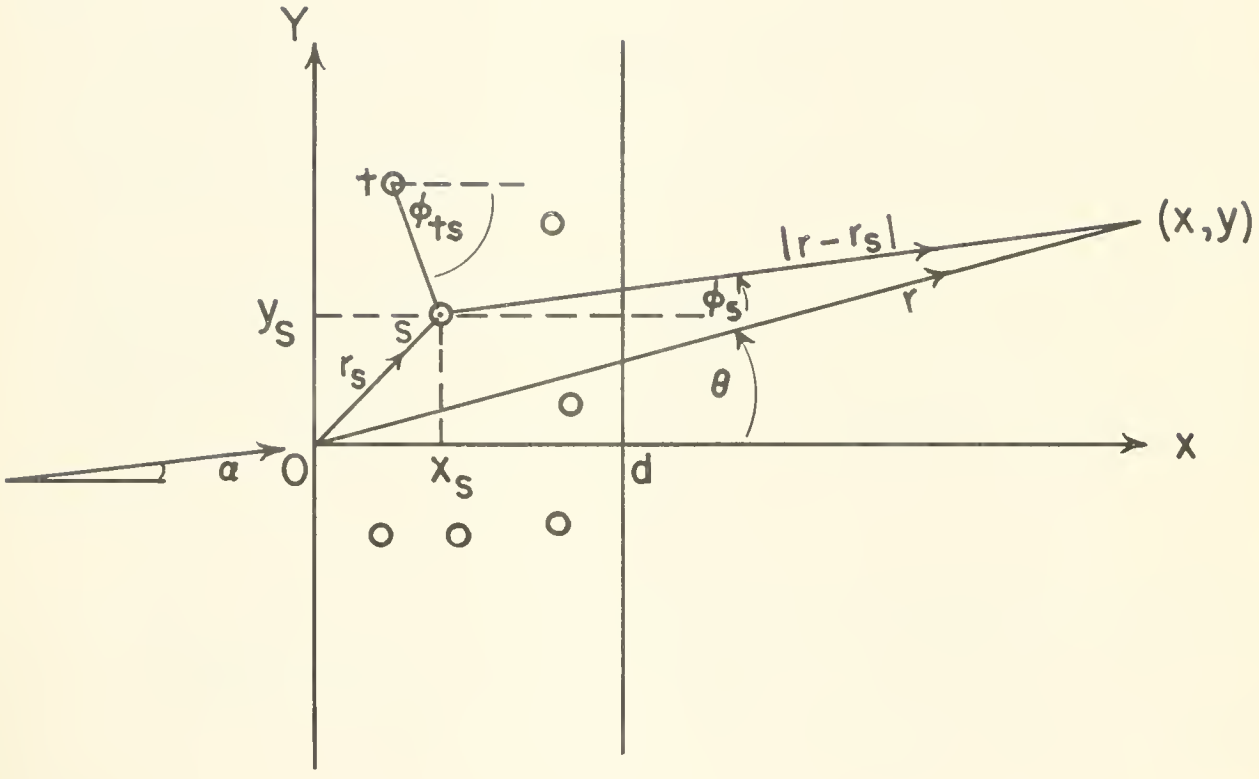

Fig. 1. Plane wave incident on a slab of parallel identical cylinders: the symbol 'r' stands for vectors.

If the plane wave is incident at angle α, i.e., if

$$(2) \qquad \psi_i = \exp(i\underline{k} \cdot \underline{r}) = \exp\left[ikr \cos(\theta - \alpha)\right] ,$$

then the wave scattered by the configuration of cylinders may be written as [1]

$$\psi = \sum_{s=1}^{N} \psi_s = \sum_{s=1}^{N} \exp(i\underline{k} \cdot \underline{r}_s) \sum_{n=-\infty}^{\infty} B_{ns}\, i^n H_n(k|\underline{r} - \underline{r}_s|) \exp(in\phi_s) ,$$

(3)

$$B_{ns} = A_n \left\{ e^{-in\alpha} + \sum_{t=1}^{N}{}' \exp\left[i\underline{k} \cdot (\underline{r}_t - \underline{r}_s)\right] \sum_{m=-\infty}^{\infty} H_{n-m}(k|\underline{r}_t - \underline{r}_s|) B_{mt} \exp\left[i(m-n)(\phi_{ts} - \tfrac{\pi}{2})\right] \right\} .$$

Here H_n is the Hankel function of the first kind. The prime on the sum over t indicates $t \neq s$.

The multiple-scattering coefficients B_{ns} depend on the single-scattering coefficients A_n (e.g., $A_n = -J_n(ka)/H_n(ka)$ for the boundary condition $\psi = 0$ at $|\underline{r} - \underline{r}_s| = a$), and on the positions of all cylinders. See ref. 1 for a list of various values of A_n for the usual electromagnetic and acoustic boundary conditions.

The Average Wave Function

A particular configuration of N cylinders, and hence a particular value of ψ, is specified by the vectors $\underline{r}_1, \underline{r}_2, \ldots \underline{r}_N$, which locate the axes of the cylinders with reference to some origin. An ensemble of configurations is specified by a distribution function $W(\underline{r}_1, \underline{r}_2, \ldots \underline{r}_N)$ such that $W\, d\underline{r}_1 \ldots d\underline{r}_N$ is the probability of finding cylinders 1 to N in their respective ranges $d\underline{r}_1$ to $d\underline{r}_N$ (i.e., cylinder 1 in the range $d\underline{r}_1$ around the point $\underline{r}_1$, etc.) The average value of $\psi(\underline{r}_1, \ldots \underline{r}_N)$ over the ensemble of configurations specified by W is thus*

$$(4) \qquad \langle \psi \rangle = \sum \langle \psi_s \rangle = \sum \int \cdots \int \psi_s(\underline{r}_1, \underline{r}_2 \ldots \underline{r}_N)\, W(\underline{r}_1, \underline{r}_2 \ldots \underline{r}_N)\, d\underline{r}_1 \ldots d\underline{r}_N$$

where the integrals are to be taken over the range available to the scatterers.

We assume that the distribution is 'uniform':

$$(5) \qquad W(\underline{r}_1, \underline{r}_2, \ldots \underline{r}_N) = P(\underline{r}_1)\, P(\underline{r}_2) \ldots P(\underline{r}_N) = V^{-N} ,$$

---

* We use either the brackets $\langle\ \rangle$ or an overhead bar to indicate an average value.

i.e., that it is equiprobable that each scatterer is anywhere in the range of integration V. The quantities V and N may approach infinity but their ratio $N/V \equiv \rho$ (where ρ is the average number of cylinders in unit range) remains constant.

Substituting $W = V^{-N}$ into (4) and integrating over all variables except $\underline{r}_s$ we obtain

$$(6) \qquad \bar{\psi} = V^{-1} \sum_s \int \exp(i\underline{k} \cdot \underline{r}_s) \sum_n \langle B_{ns} \rangle_s \, i^n H_n(k|\underline{r} - \underline{r}_s|) \exp(in\phi_s) \, d\underline{r}_s \quad ,$$

where

$$(6') \qquad \langle B_{ns} \rangle_s = A_n \Big\{ e^{-in\alpha} + V^{-1} \sum_t{}' \int \exp\left[i\underline{k} \cdot (\underline{r}_t - \underline{r}_s)\right] \sum_m \langle B_{mt} \rangle_{st} \cdot H_{n-m}(k|\underline{r}_t - \underline{r}_s|) \exp\left[i(n-m)(\phi_{ts} - \tfrac{\pi}{2})\right] d\underline{r}_t \Big\}$$

is the average value of B_{ns} over all variables not equal to s, i.e., the average with one variable held fixed; similarly $\langle B_{nt} \rangle_{st}$ is the corresponding average with two fixed variables.

We restrict discussion to an infinite slab of finite thickness, i.e., to the range of integration $y_s = -\infty$ to $+\infty$ and $x_s = 0$ to d. From physical considerations, or by inspection of (6'), we see that for this case $\langle B_{ns} \rangle_s$ is independent of y_s, i.e., the average scattering coefficients depend only on the distances of the scatterer from the interface. We may thus integrate immediately over y_s in (6). Introducing the abbreviations $h = k \cos\alpha$ and $p = k \sin\alpha$ we obtain

$$(7) \qquad \bar{\psi} = e^{ipy} V^{-1} \sum_s \int_0^d \exp(ihx_s) \sum_n \langle B_{ns}(x_s) \rangle_s \, I(x_s - x) \quad ,$$

where

$$I(x_s - x) = i^n \int_{-\infty}^{\infty} e^{ip(y_s - y)} H_n\left(k\sqrt{(x_s - x)^2 + (y_s - y)^2}\right) \exp\left[in \tan^{-1} \frac{y_s - y}{x_s - x}\right] dy_s \quad ,$$

$$(8) \qquad = \frac{2}{h} \exp(ih|x_s - x|) \exp\left(in \begin{bmatrix} \alpha \\ \pi - \alpha \end{bmatrix}\right), \quad \begin{bmatrix} x > x_s \\ x < x_s \end{bmatrix} \quad .$$

See reference 1 for details of this evaluation; we have used an identity established by F. Reiche[5]. $\bar{\psi}$ is thus a 'sum' of plane waves: the waves originat-

ing at $x_s \lessgtr x$ are respectively the average transmitted and reflected waves arising from the planes of scatterers.

To proceed further we introduce the approximation

$$(9) \qquad \langle B_{mt} \rangle_{st} \approx \langle B_{mt} \rangle_t \equiv \bar{B}_m(xt) \ ,$$

i.e., we will approximate the scattering coefficients with two variables held fixed by the result for one fixed variable. (See reference 1 for a discussion of the physical implications of this approximation.) Thus we approximate (6) by

$$\bar{B}_n(x_s) = A_n \left\{ e^{-in\alpha} + (\rho/N) \sum_t{}' \int_o^d \exp\left[ih(x_t - x_s)\right] \sum_m \bar{B}_m(x_t) \ I'(x_t - x_s) \right\};$$

$$I'(x_t - x_s) = i^{-n+m} \int_{-\infty}^{\infty} e^{ip(y_t - y_s)} H_{n-m}\left(k \sqrt{(x_t-x_s)^2+(y_t-y_s)^2}\right) \exp\left[-i(n-m)\tan^{-1}\left\{\frac{y_t-y_s}{x_t-x_s}\right\}\right]$$

$$(10) \qquad = \frac{2}{h} \exp(ih|x_t-x_s|) \exp\left(i(m-n) \begin{bmatrix} \alpha \\ \pi - \alpha \end{bmatrix}\right), \quad \begin{bmatrix} x_s > x_t \\ x_s < x_t \end{bmatrix} \ .$$

We now sum over s and t; for the second we replace (N-1)/N by unity. Introducing the new symbols u and v for x_s and x_t we have

$$(11) \qquad \bar{\psi} = e^{ipy} \left(\frac{2\rho}{h}\right) \int_o^d e^{ihu + ih|u-x|} \sum_n \bar{B}_n(u) \exp\left(in \begin{bmatrix} \alpha \\ \pi - \alpha \end{bmatrix}\right) du \ , \quad \begin{bmatrix} x > u \\ x < u \end{bmatrix}$$

$$\bar{B}_n(u) = A_n \left\{ e^{-n\alpha} + \exp\left(in \begin{bmatrix} -\alpha \\ -\pi + \alpha \end{bmatrix}\right) \frac{2\rho}{h} \int_o^d e^{ih(v-u)+ih|v-u|} \right.$$

$$\left. \cdot \sum_m \bar{B}_m(v) \exp\left(im \begin{bmatrix} \alpha \\ \pi - \alpha \end{bmatrix}\right) dv \right\}, \quad \begin{bmatrix} u > v \\ u < v \end{bmatrix}.$$

We introduce the following definitions:

$$C(\phi, \alpha) \equiv \sum A_n e^{in(\phi - \alpha)} \ , \quad Z(\phi, \alpha) = (2\rho/h) \ C(\phi, \alpha) \ ;$$

$$(12) \qquad G(u; \phi, \alpha) \equiv \sum \bar{B}_n \ C^{in\phi} \ , \quad g(u; \phi, \alpha) = (2\rho/h) \ e^{ihu} \ G(u; \phi, \alpha) \ .$$

Here $C(\phi, \alpha)$, the single-scattered amplitude of an isolated cylinder, is a known function prescribed by the boundary conditions. $Z(\phi, \alpha)$ is the single-scattered amplitude for a plane of cylinders. $G(u; \phi, \alpha)$ is the average multiple-scattered amplitude of a cylinder at $x = u$. Finally $g(u; \phi, \alpha)e^{ipy}$ with $\phi = \alpha$ or $\pi - \alpha$ is the incident wave times the transmitted or reflected amplitude of the plane of cylinders at $x = u$.

We rewrite (11) in terms of (12) and add ψ_i to obtain the total wave function:

$$\psi_i + \bar{\psi} = e^{ipy+ihx}\left[1 + \int_0^x e^{-ihu}\, g(u;\, \alpha,\alpha)du\right] + e^{ipy-ihx}\int_x^d e^{ihu} g(u;\pi-\alpha,\alpha)\, du$$

(13)

$$\equiv \psi_i + \bar{\psi}_+(0,x) + \bar{\psi}_-(x,d),$$

where

$$g(u;\phi,\alpha) = Z(\phi,\alpha)e^{ihu}\left[1 + \int_0^u e^{-ihv} g(v;\alpha,\alpha)dv\right] + Z(\phi,\pi-\alpha)e^{-ihu}\int_u^d e^{ihv} g(v;\pi-\alpha,\alpha)\, dv$$

(14)

$$= e^{-ipy} Z(\phi,\alpha)\left[\psi_i(u) + \bar{\psi}_+(0,u)\right] + e^{-ipy}\, Z(\phi,\pi-\alpha)\, \bar{\psi}_-(u,d).$$

$\psi_i + \bar{\psi}$ of (13) is the 'internal' wave; i.e., (13) holds for $0 < x < d$. If $x > d$, then $\psi_i + \bar{\psi}_+(0,d)$ is the 'transmitted' wave; if $x < 0$, then $\bar{\psi}_-(0,d)$ is the 'reflected' wave. Note that $(0,d)$ in $\bar{\psi}_+(0,d)$ merely indicates the limits of integration; the result is still a function of x.

The field within the slab at u thus consists of essentially three components: the incident wave ψ_i; the 'sum' of transmitted waves $\bar{\psi}_+$ arriving from the planes of scatterers at $0 < u < x$; and the sum of reflected waves $\bar{\psi}_-$ arriving from $x < u < d$. Note that g of (14) is not simply proportional to $\psi_i + \bar{\psi}$; instead, g consists of essentially two components - the term in $Z(\phi,\alpha)$ arising from all transmitted waves which are incident at angle α, and the term in $Z(\phi,\pi-\alpha)$ arising from the reflected waves which are incident at $\pi - \alpha$. The integral relation (14) may be solved directly for g: we substitute $\phi = \alpha, \pi-\alpha$ into (14) and obtain two simultaneous integral equations which we solve by inspection.

We let $g(u; \phi, \alpha) = A(\phi)\, e^{i\gamma u} + B(\phi)\, e^{i\delta u}$, where A, B, γ and δ are independent of u; we abbreviate $A(\alpha)$ and $A(\pi - \alpha)$ by A_+ and A_- respectively, and similarly for $B(\phi)$. Substituting into (14) and evaluating the integrals over v we find

$$g(u;\phi,\alpha) = A(\phi)e^{i\gamma u} + B(\phi)\, e^{i\delta u}$$

$$(15) \qquad = e^{ihu}Z(\phi,\alpha)\left[1 - \frac{A_+}{i(\gamma-h)} - \frac{B_+}{i(\delta-h)}\right] + e^{ih(d-u)}Z(\phi,\pi-\alpha)\left[\frac{A_-e^{i\delta d}}{i(\gamma+h)} + \frac{B_-e^{i\delta d}}{i(\delta+h)}\right]$$

$$+ e^{i\gamma u}\left[\frac{A_+Z(\phi,\alpha)}{i(\gamma-h)} - \frac{A_-Z(\phi,\pi-\alpha)}{i(\gamma+h)}\right] + e^{i\delta u}\left[\frac{B_+Z(\phi,\alpha)}{i(\delta-h)} - \frac{B_-Z(\phi,\pi-\alpha)}{i(\delta+h)}\right] .$$

Now we substitute $\phi = \alpha,\ \pi-\alpha$ into (15) and introduce the symbols

$$(16) \quad T = Z(\alpha,\alpha) = Z(\pi-\alpha,\pi-\alpha) = \frac{2\rho}{h}C(\alpha,\alpha),\ R = Z(\alpha,\pi-\alpha) = Z(\pi-\alpha,\alpha) = \frac{2\rho}{h}C(\pi-\alpha,\alpha)\ ;$$

then, equating the coefficients of $e^{\mp ihu}$, $e^{i\gamma u}$ and $e^{i\delta u}$ to zero, we obtain six simultaneous equations for the six unknowns $A_\pm, B_\pm$, γ and δ:

$$a)\ 1 - \frac{A_+}{i(\gamma-h)} - \frac{B_+}{i(\delta-h)} = 0\ , \qquad b)\ \frac{A_-e^{i\gamma d}}{i(\gamma+h)} + \frac{B_-e^{i\delta d}}{i(\delta+h)} = 0\ ,$$

$$(17) \quad c)\ A_+ = \frac{A_+T}{i(\gamma-h)} - \frac{A_-R}{i(\gamma+h)}\ , \qquad d)\ B_+ = \frac{B_+T}{i(\delta-h)} - \frac{B_-R}{i(\delta+h)}\ ,$$

$$e)\ A_- = \frac{A_+R}{i(\gamma-h)} - \frac{A_-T}{i(\gamma+h)}\ , \qquad f)\ B_- = \frac{B_+R}{i(\delta-h)} - \frac{B_-T}{i(\delta+h)}\ .$$

Solving equations (17) we obtain

$$(18) \qquad \gamma = -\delta = \left[(h - iT)^2 + R^2\right]^{1/2} = \left[h^2 - i2hT - T^2 + R^2\right]^{1/2}$$

$$(19) \quad \begin{cases} A_+ = i(\gamma - h)D\ , & B_+ = q^2\, i(\gamma + h)\, e^{i2\gamma d}\, D\ , \\ A_- = qi(\gamma + h)D\ , & B_- = qi\,(\gamma - h)\, e^{i2\gamma d}\, D\ , \\ q^2 = \dfrac{T+ih-i\gamma}{T+ih+i\gamma} = \dfrac{R^2}{(T+ih+i\gamma)^2} = \dfrac{(T+ih-i\gamma)^2}{R^2}\ , & D^{-1} = 1 - q^2\, e^{i2\gamma d}\ . \end{cases}$$

The multiple-scattered amplitude for a cylinder at u (i.e., $G = (h/2\rho)e^{-ihu}g$) is thus

$$(20) \quad G(u;\phi,\alpha) = e^{i(\gamma-h)u}\left[C(\phi,\alpha) - qC(\phi,\pi-\alpha)\right]D + e^{-i(\gamma+h)u+i2\gamma d}q\left[C(\phi,\pi-\alpha) - qC(\phi,\alpha)\right]D\ .$$

Referring back to (13) we make use of (18) and (19) to obtain

$$\psi_i + \bar{\psi}_+(0,x) = \psi_i\left[e^{i(\gamma-h)x} - q^2 e^{-i(\gamma+h)x+i2\gamma d}\right]D \quad , \quad \psi_i = e^{ipy+ihx} \; ;$$

(21)

$$\bar{\psi}_-(x,d) = -q\, e^{ipy-ihx}\left[e^{i(\gamma+h)x} - e^{-i(\gamma-h)x+i2\gamma d}\right] D \quad .$$

Consequently the internal, transmitted, and reflected waves are given respectively by

$$(22)\quad \bar{\psi}_I = \psi_i + \bar{\psi}_+(0,x) + \bar{\psi}_-(x,d) = (1-q)e^{ipy}\,\frac{\left[e^{i\gamma x} + q\, e^{-i\gamma(x-2d)}\right]}{1 - q^2 e^{i2\gamma d}} \quad ,$$

$$(23)\quad \bar{\psi}_T = \psi_i + \bar{\psi}_+(0,d) = \psi_i\,\frac{(1-q^2)\, e^{i(\gamma-h)d}}{1 - q^2\, e^{i2\gamma d}} \quad ,$$

$$(24)\quad \bar{\psi}_R = \bar{\psi}_-(0,d) = -e^{ipy-ihx}\,\frac{q(1 - e^{i2\gamma d})}{1 - q^2\, e^{i2\gamma d}} \quad .$$

If $d \to \infty$, i.e., if we have an infinite half-space of scatterers, then

$$(25)\quad \bar{\psi}_I \to (1 - q)e^{i\gamma x+ipy} \quad , \quad \bar{\psi}_T \to 0 \quad , \quad \bar{\psi}_R \to -q\, e^{-ihx+ipy} \quad .$$

Our results are formally identical with those obtained for a uniform slab with propagation constant

$$(26)\quad K = \left[\gamma^2 + p^2\right]^{1/2} = \left[k^2 - i2k\cos\alpha\, T - T^2 + R^2\right]^{1/2}$$

subject to the continuity boundary conditions

$$(27)\quad \psi_0 = F\,\psi_I \quad , \quad \frac{\partial\psi_0}{\partial x} = M\,\frac{\partial\psi_I}{\partial x} \quad , \quad x = 0 \text{ and } d \quad ,$$

where 0 and I stand for 'outside' and 'inside' and F and M are the constants

$$(28)\quad F = 1 \quad , \quad M = \frac{(1 + q)h}{(1 - q)\gamma} = \frac{ih}{T - R + ih} \quad .$$

The equivalent physical parameters of our 'uniform slab' may be determined by comparsion with the usual acoustic and electromagnetic boundary conditions. Thus if ψ is the acoustic velocity potential, then $M/F = \rho_0/\rho_I$ is the ratio of

the material densities. For the electromagnetic case of E perpendicular to the plane of incidence (E $\propto\psi$), the ratio of permeabilities is $M/F = \mu_0/\mu_I$; for E parallel to the plane of incidence (H $\propto\psi$), $M/F = \mu_I h^2/\mu_0\gamma^2$. Note that for the internal field, H $\propto (K/\mu_I)E$, where K is given by (26).

Rewriting $\exp(i\gamma x + ipy)$ as $\exp\left[-\mathrm{Im}(\gamma x)+i(\mathrm{Re}(\gamma x)+k \sin\alpha\, y)\right]$ where Im γ is the 'absorption' coefficient, we see that the surfaces of constant amplitude are planes parallel to the face of the slab. The surfaces of constant phase are given by

$$\mathrm{Re}(\gamma x) + k \sin\alpha\, y \equiv k \cos\beta\, x + k \sin\beta y = \text{constant},$$

where β is the real angle of refraction. Thus

$$(29)\quad \cos\beta = \frac{\mathrm{Re}\gamma}{\left[(\mathrm{Re}\gamma)^2 + k^2\sin^2\alpha\right]^{1/2}}\ , \quad \sin\beta = \frac{k \sin\alpha}{\left[(\mathrm{Re}\gamma)^2 + k^2\sin^2\alpha\right]^{1/2}}\ ,$$

and the real index of refraction of the slab is

$$(30)\quad \nu(\alpha) = \frac{\sin\alpha}{\sin\beta} = \left[(\mathrm{Re}\ \gamma/k)^2 + \sin^2\alpha\right]^{1/2}\ ;$$

note that ν is an explicit function of the angle of incidence (as for a uniform slab of conducting dielectric).

For the special case of 'isotropic scatterers' (i.e., when $C(\phi,\alpha)$ is independent of ϕ and α), we have T = R and consequently

$$(31)\quad \gamma = \left[h^2 - i2hT\right]^{1/2}\ , \quad q = (\gamma - h)/(\gamma + h)\ .$$

For this case our results are formally identical with those obtained by Reiche [2] and by Foldy[3]. More generally, however, γ and q as given by (18) and (19) are explicit functions of the angle of incidence. For the general case our results differ from those of Lax[4] in that his γ depends only on the forward scattered amplitude of the isolated scatterer, i.e., on $C(\alpha,\alpha)$ as in $T = (2\rho/h)C(\alpha,\alpha)$, while our γ depends also on the scattering amplitude in the direction of specular reflection with respect to the face of the slab, i.e., on $C(\pi - \alpha,\alpha)$ as in R.

The Average Intensity

The average intensity is obtained by averaging $|\psi|^2 = \sum\sum \psi_s \psi_t^* =$

$= \sum\sum{}' \psi_s \psi_t^* + \sum |\psi_s|^2$ (where ψ is given by (3)) over the ensemble specified by W of (5). We find in general that we may write

$$(32) \qquad \langle|\psi|^2\rangle = |\bar{\psi}|^2 + f \ ,$$

where $|\bar{\psi}|^2$, the 'coherent intensity', is obtained from the results of the preceding section. The component f, the 'incoherent intensity', arises primarily from the terms s = t of $|\psi|^2$; it is a complicated function depending on the value of $\langle G(x;\phi,\alpha)\, G^*(x';\phi',\alpha)\rangle - \bar{G}(x;\phi,\alpha)\bar{G}^*(x';\phi',\alpha)$. We have found no elegant method of deriving f in closed form.

We obtain an approximation of f by following a heuristic procedure based on Foldy's analysis of isotropic scatterers.[3] From (3) we write the average multiple scattered wave arising from cylinder s as

$$(33) \qquad \bar{\psi}_s = \sum_n i^n \bar{B}_n(x_s) \exp(in\phi_s)\, H_n(k|\underline{r} - \underline{r}_s|) \ .$$

Using the asymptotic representation $H_n(x) \sim (2/\pi x)^{1/2} e^{ix} i^{-n}$, we obtain the far-field form of (33):

$$(34) \qquad \bar{\psi}_s \sim \left[\frac{2}{\pi k|\underline{r}-\underline{r}_s|}\right]^{1/2} \exp(ik|\underline{r}-\underline{r}_s|)\sum_n \bar{B}_n(x_s)\exp(in\phi_s)$$

$$= \left[\frac{2}{\pi k|\underline{r}-\underline{r}_s|}\right]^{1/2} \exp(ik|\underline{r}-\underline{r}_s|)\, G(x_s;\phi_s,\alpha) \ ,$$

where G is given by (20). We may use (34) to obtain a first approximation for $\langle|\psi_s|^2\rangle$, i.e., we may simply use $\langle|\bar{\psi}_s|^2\rangle \approx |\bar{\psi}_s|^2$. However, although this takes into account the fact that the excitation of s is no longer simply the incident wave ψ_i, it does not take into account the 'shielding' effects of the cylinders between s and the observation point.

In order to take shielding into account we modify the exponent of (34): we replace k (the free-space propagation constant) by K (the value inside the equivalent homogeneous slab); also we replace the path distance $|\underline{r} - \underline{r}_s|$ by that part of the distance which lies within the slab, say R_s. Thus we write

$$(35)\quad \langle|\psi_s|^2\rangle \approx \left[\frac{2}{\pi k|\underline{r}-\underline{r}_s|}\right] \exp\left[-2\mathrm{Im}(K)R_s\right]\ |G(x_s;\phi_s\alpha)|^2\ ,$$

where $R_s = |(x - x_s)\sec\phi_s|$ if $0 < x < d$; for $x < 0$ and $x > d$ we replace x in R_s by 0 and d respectively.

We now approximate f of (32) by

$$(36)\quad f \approx \frac{2\rho}{\pi k}\int_{-\infty}^{\infty} dv \int_0^d du\ e^{-2\mathrm{Im}(K)|(x-u)\sec\phi|}\ \frac{|G(u;\phi,\alpha)|^2}{\left[(x-u)^2 + (y-v)^2\right]^{1/2}},\quad \phi = \tan^{-1}\left[\frac{y-v}{x-u}\right]\ .$$

(36) corresponds to f_I; f_R and f_T are obtained by replacing x by 0 and d respectively. Our result for f corresponds essentially to the leading term of the rapidly converging series derived by Foldy.

By means of the transformation

$$du\,dv = -du\left[(x-u)^2 + (y-v)^2\right]^{1/2}\ \sec\phi$$

we can replace the integrations over v in (36) by an integration over ϕ; we obtain

$$(37)\quad f = \frac{2\rho}{\pi k}\left[\int_{-\pi/2}^{\pi/2}\int_0^x - \int_{\pi/2}^{3\pi/2}\int_x^d\right] e^{-2\mathrm{Im}(K)(x-u)\sec\phi}\ \frac{|G(u;\phi)|^2}{\cos\phi}\ d\phi\,du\ ,$$

where the integrations over u may be performed directly if we substitute $A(\phi)e^{i\gamma u} + B(\phi)e^{-i\gamma u}$ for $G(u,\phi)$. The integrals over ϕ may be evaluated for special values of $C(\phi,\alpha)$.

The form of (37) holds for $0 < x < d$; for $x < 0$ or $x > d$ we let $x = 0$ or $x = d$ rsspectively to obtain the appropriate values of f.

The Average Energy Flux

In this Section we obtain expression for the reflection and transmission coefficients, for the scattering cross section, and for other functions which are of particular physical interest.

The average energy flux, i.e., the ensemble average of the time-averaged power flow through unit area of wavefront, is in general proportional to $\mathrm{Re}(A\,\psi\,\nabla\psi^*)$; for acoustics A equals the undisturbed density of the medium while for electromagnetics A equals K/μ . We choose a normalization such that the incident energy flux equals unity. We will not present an expression for

the flux for the general case, but rather state only the functions of particular physical interest.

The 'reflection coefficient' may be defined as the ratio of the coherently reflected flux normal to the face of the slab and the incident flux normal to to the slab:

$$(38)\quad \mathcal{R} = \frac{\mathrm{Re}(\vec{\psi}_R^{*}\,\nabla\,\bar{\psi}_R)\cdot\underline{i}}{\mathrm{Re}\,\psi_i^{*}\,\nabla\,\psi_i\cdot\underline{i}} = -\mathrm{Re}\left[\vec{\psi}_R^{*}\,\frac{\partial\,\psi_R}{\partial(ihx)}\right] = -\left|\frac{q(1-e^{i2\gamma d})}{1-q^2e^{i2\gamma d}}\right|^2$$

where $\underline{i}$ is the unit vector normal to the surface . The total normalized coherent flux normal to the surface for $x < 0$ is thus $(1 - \mathcal{R})\cos\alpha$.

Similarly the 'transmission coefficient' is given by

$$(39)\qquad \mathcal{T} = \mathrm{Re}\left[\vec{\psi}_T^{*}\,\frac{\partial\bar{\psi}_T}{\partial(ihx)}\right] = e^{-2\mathrm{Im}(\gamma d)}\left|\frac{1-q^2}{1-q^2e^{i2\gamma d}}\right|^2,$$

and the normalized flux to the surface is $\mathcal{T}\cos\alpha$.

Inside the slab we define the "propagation coefficient"

$$\mathcal{P}(x) \equiv \mathrm{Re}\left[\vec{\psi}_I^{*}\,M\,\frac{\partial\,\bar{\psi}_I}{\partial(ihx)}\right]$$

$$(40)\qquad = \left|\frac{1-q}{1-q^2\,e^{i2\gamma d}}\right|^2 \mathrm{Re}\left\{\frac{\left[e^{i\gamma x}+q\,e^{-i\gamma(x-2d)}\right]^{*} i\gamma\left[e^{i\gamma x}-q\,e^{-i\gamma(x-2d)}\right]}{T-R+ih}\right\}$$

where M was obtained from (28). $\mathcal{P}\cos\alpha$ is thus the normalized coherent flux in the interior normal to the faces of the slab. It is clear that the coherent flux is conserved on passage through the slab; i.e., $(1 - R) = \mathcal{P}(0)$ and $\mathcal{P}(d) = \mathcal{T}$. However, we must show that the normal components of the total flux are continuous across the interfaces $x = 0$, d.

Since $\nabla\psi_s = ik\psi_s \underline{r}_s$, where $\underline{r}_s = \underline{i}\cos\phi + \underline{j}\sin\phi$ is a unit vector from s to the observation point, we assume that the incoherent flux may be written in the form

$$\underline{I} = f(\underline{r}_s) \tag{41}$$

where $f(\underline{r}_s)$ indicates that $\underline{r}_s$ is to be inserted within the integrals of (36). From the forms of (36) and (41) it is clear that $\underline{I} \cdot \underline{i} = f(\cos\phi)$ is continuous across $x = 0$ and $x = d$. Consequently the total flux (coherent plus incoherent) through unit area of the interfaces is continuous, as is required by the energy principle.

We now state certain energy theorems and use these to check our results in detail. Specifically we investigate the relationship between our heuristic approximation for the incoherent flux and our relatively rigorous expressions for the coherent flux (i.e., rigorous except for the approximation (9)). For simplicity we assume that the cylinders are non-absorbing (i.e., either perfectly conducting or perfectly dielectric); also we neglect Foldy's 'shielding factor' $\mathrm{Im}(K)\sec\phi$ in (36). The following may of course be extended to the case of absorbing cylinders by means of the procedure of reference 1. We denote $f(\cos\phi)$ without the shielding factor by I.

If the cylinders are non-absorbing, then the total normal flux for $x < 0$ must equal that for $x > d$, i.e.,

$$(1 - \mathcal{R})\cos\alpha + I_R = \mathcal{T}\cos\alpha + I_T \ . \tag{42}$$

Regrouping the terms yields the general theorem

$$\begin{aligned} 1 - \mathcal{R} - \mathcal{T} = \mathcal{P}(0) - \mathcal{P}(d) &= \sec\alpha\,(I_T - I_R) \\ &= \frac{2\rho}{\pi h}\left[\int_{-\pi/2}^{\pi/2} + \int_{\pi/2}^{3\pi/2}\right]\int_0^d |G(u;\phi)|^2\,du\,d\phi = \frac{\rho}{\cos\alpha}\int_0^d Q(u)\,du \ , \end{aligned} \tag{43}$$

where

$$Q(u) = \frac{2}{\pi k}\int_0^{2\pi} |G(u;\phi)|^2\,d\phi = \int_0^{2\pi} \Delta Q(u,\phi)\,d\phi \tag{44}$$

is the average multiple-scattered cross section for a cylinder at u, and $\Delta Q(u;\phi)$ is the differential scattering cross section. Equation (43) states: <u>the energy</u>

'removed' from unit area of the incident wavefront by reflection and transmission (or equivalently, the coherent flux entering the slab minus that leaving the slab) equals the total energy scattered by the elements 'illuminated' by unit area of wavefront (i.e., by the scatterers in the volume 1 sec $\alpha \cdot d$).

Similarly the energy principle requires that within the slab the total normal flux at x equals the flux at $x + \Delta x$, i.e., that

$$(45)\qquad \mathcal{P}(x)\cos\alpha + I(x) = (x+\Delta x)\cos\alpha + I(x+\Delta x),$$

or

$$(46)\qquad \Delta\mathcal{P} = \mathcal{P}(x+\Delta) - \mathcal{P}(x) = -\sec\alpha\,[I(x+\Delta x) - I(x)] = -\sec\alpha\,\Delta I$$

$$= -\frac{\rho}{\cos\alpha}\int_x^{x+\Delta x} Q(u)\,du\ .$$

We will demonstrate that our expressions for $\mathcal{P}$ and Q satisfy this relation; for simplicity we treat the case $d\to\infty$ (the extension to finite d is straightforward).

For $d\to\infty$ we have

$$(47)\qquad \Delta\mathcal{P} = \mathcal{P}(x)\left[e^{-2\mathrm{Im}\gamma\Delta x}-1\right],\ \mathcal{P}(x) = |1-q|^2\,\mathrm{Re}\left[\frac{i\gamma}{T-R+ih}\right]e^{-2\mathrm{Im}\gamma x} = (1-|q|^2)e^{-2\mathrm{Im}\gamma x}$$

and

$$(48)\qquad -\sec\Delta I = -\frac{2\rho}{\pi h}\int_x^{x+\Delta x} e^{-2\mathrm{Im}\gamma u}\,du\,J\ ,$$

$$J = \int_0^{2\pi} |C(\phi,\alpha) - q\,C(\phi,\pi-2)|^2\,d\phi$$

$$= \int_0^{2\pi} \sum_n\sum_m A_n A_m^* e^{i(n-m)\phi}\left[e^{-in\alpha} - qe^{-in(\pi-2)}\right]\left[e^{im\alpha} - q^* e^{im(\pi-2)}\right]\,du$$

$$= 2\pi\sum |A_n|^2\left[1 + |q|^2 - (q+q^*)\cos n(\pi-2\alpha)\right]\ .$$

Since the cylinders are non-absorbing we have† $|A_n|^2 = -\mathrm{Re}(A_n)$. Consequently $\sum |A_n|^2 = -\mathrm{Re}\sum A_n = -\mathrm{Re}\, C(\alpha,\alpha)$; similarly $\sum |A_n|^2 \cos n(\pi-2\alpha) = -\mathrm{Re}\, C(\alpha,\pi-\alpha)$. Hence (48) simplifies to

$$-\sec\alpha\,\Delta I = 2\mathrm{Re}\left|T(1+|q|^2) - 2\mathrm{Re}(q)R\right| \int_x^{x+\Delta x} e^{-2\mathrm{Im}(\gamma)u}\,du$$

(49)

$$= -\mathrm{Re}\left[T(1+|q|^2) - 2\mathrm{Re}(q)R\right] e^{-2\mathrm{Im}\gamma x}\,\frac{\left[e^{-2\mathrm{Im}\gamma\Delta x} - 1\right]}{\mathrm{Im}(\gamma)}\,.$$

Substituting (47) and (49) into (46) we obtain

$$(50)\qquad 1 - |q|^2 = \frac{\mathrm{Re}\left[T(1+|q|^2) - 2\mathrm{Re}(q)R\right]}{\mathrm{Im}\,\gamma}\,.$$

We must now show that this equality holds.

From (19) we have $q = (T + ih - i\gamma)/R = R/(T + ih + i\gamma)$; consequently $\mathrm{Re}(Rq) = \mathrm{Re}T + \mathrm{Im}\gamma$, $\mathrm{Re}(Rq^*) = (\mathrm{Re}T - \mathrm{Im}\gamma)|q|^2$. Substituting

$$2\mathrm{Re}(R)\mathrm{Re}(q) = \mathrm{Re}(Rq) + \mathrm{Re}(Rq^*) + \mathrm{Re}T(1+|q|^2) + \mathrm{Im}\gamma(1-|q|^2)$$

into (50), we see that the right side reduces to $1 - |q|^2$. Hence the equality (50) holds and our heuristic expression for the incoherent flux satisfies the energy principle.

For practical applications we reinsert the shielding factor $\mathrm{Im}(K)\sec\phi$ and write the incoherent scattering for $x < 0$, $d \to \infty$, as

$$(51)\qquad \underline{I} = -\frac{2\rho}{\pi k}\int_{\pi/2}^{3\pi/2}\int_0^{\infty} \exp(-2\mathrm{Im}(\gamma)u + 2\mathrm{Im}(K)u\sec\phi)\,\frac{|C(\phi,\alpha) - qC(\phi,\pi-\alpha)|^2}{\cos\phi}\,\underline{r}(\phi)\,d\phi\,dx.$$

Integrating over x we obtain

$$\underline{I} = \frac{2\rho}{\pi k}\int_{\pi/2}^{3\pi/2} \frac{|C(\phi,\alpha) - qC(\phi,\pi-\alpha)|^2\,\underline{r}}{2\cos\phi\left[-\mathrm{Im}(\gamma) + \mathrm{Im}(K)\sec\phi\right]}\,d\phi$$

(52)

$$\equiv \int_{\pi/2}^{3\pi/2} J(\phi)\,\underline{r}(\phi)\,d\phi = \int_{-\pi/2}^{\pi/2} J(\mu)\,\underline{r}(\mu)\,d\mu\,,$$

---

† For circular cylinders this follows from the form of A_n; e.g., if $A_n = -J_n/(J_n + iN_n)$ $= -(J_n^2 + iJ_nN_n)/(J_n^2+N_n^2)$, then clearly $\mathrm{Re}(A_n) = -|A_n|^2$. More generally the single-scattered energy theorem yields[6] $\mathrm{Re}C(\alpha,\alpha) = -\frac{1}{2\pi}\int_0^{2\pi}|C(\phi,\alpha)|^2 d\phi$ for non-absorbing scatterers. This expression reduces to $\mathrm{Re}\left(\sum_{-\nu}^{\nu} A_n\right) = -\sum_{-\nu}^{\nu}|A_n|^2$ provided that the scattering amplitude may be written as the Fourier series $C(\phi,\alpha) = \sum_{-\nu}^{\nu} A_n e^{in(\phi-\alpha)}$.

where $\mu = \pi - \phi$ is measured from the negative x- axis. The function

$$(53) \qquad J(\mu) = \frac{\rho}{\pi k} \frac{|C(\pi-\mu,\alpha) - qC(\pi-\mu,\pi-\alpha)|^2}{\cos\mu\ \mathrm{Im}(Y) + \mathrm{Im}(K)}$$

is the energy scattered into unit solid angle at an angle μ with the surface normal; i.e., $J(\mu)$ is the differential cross section for the entire distribution (as measured on the 'reflection side' of the infinitely thick slab). $J(\mu)$ is the scattering function of primary physical interest.

Finally we consider the behavior of the distribution at grazing incidence ($\alpha \to \pi/2$) and show that our present results are in accord with those of reference 1 and with known experiments. For $\alpha \to \pi/2$, we have $C(\phi,\alpha) \to C(\phi, \pi-\alpha)$; furthermore from (16), (18) and (19) we find that Y and K are finite and that $q \to 1$ as $\alpha \to \pi/2$. Consequently (53) vanishes for all angles μ at grazing incidence. From this we deduce that the variance of ψ (the mean square fluctuation) is zero at $\alpha \to \pi/2$ and that $\bar{\psi} \to \psi$ is sufficient in itself to characterize the effects of the distribution (i.e., the intensity is all 'coherent'). From (22), (23) and (24) we see that for $\alpha \to \pi/2$, both $\bar{\psi}_I$ and $\bar{\psi}_T$ vanish while $\bar{\psi}_R$ reduces simply to $-\psi_i$. Thus the distribution acts like a perfect dielectric at grazing incidence; see reference 1 for further discussion.

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